The correspondence of some orthogonal series coefficient arrays. R. Rogers -Oct 21 2014

The result is $*1$ below.

Let: $[A] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ a_{10} & 1 & 0 & 0 & 0 \\ a_{20} & a_{21} & 1 & 0 & 0 \\ a_{30} & a_{31} & a_{32} & 1 & 0 \\ a_{40} & a_{41} & a_{42} & a_{43} & 1 \end{bmatrix}$

$[B] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ b_{10} & 1 & 0 & 0 & 0 \\ b_{20} & b_{21} & 1 & 0 & 0 \\ b_{30} & b_{31} & b_{32} & 1 & 0 \\ b_{40} & b_{41} & b_{42} & b_{43} & 1 \end{bmatrix}$

Temporarily tossing the main diagonal.

$[A'] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ a_{10} & 0 & 0 & 0 & 0 \\ a_{20} & a_{21} & 0 & 0 & 0 \\ a_{30} & a_{31} & a_{32} & 0 & 0 \\ a_{40} & a_{41} & a_{42} & a_{43} & 0 \end{bmatrix}$

$[B'] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ b_{10} & 0 & 0 & 0 & 0 \\ b_{20} & b_{21} & 0 & 0 & 0 \\ b_{30} & b_{31} & b_{32} & 0 & 0 \\ b_{40} & b_{41} & b_{42} & b_{43} & 0 \end{bmatrix}$

If A', B' are maximal rank nilpotent then they are equivalent (in the matrix sense) to the Jordan form like so:

$$[U_a] \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ a_{10} & 0 & 0 & 0 & 0 \\ a_{20} & a_{21} & 0 & 0 & 0 \\ a_{30} & a_{31} & a_{32} & 0 & 0 \\ a_{40} & a_{41} & a_{42} & a_{43} & 0 \end{bmatrix} [U_a]^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = [J]$$

$$[U_b] \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ b_{10} & 0 & 0 & 0 & 0 \\ b_{20} & b_{21} & 0 & 0 & 0 \\ b_{30} & b_{31} & b_{32} & 0 & 0 \\ b_{40} & b_{41} & b_{42} & b_{43} & 0 \end{bmatrix} [U_b]^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = [J]$$

In addition we see that

$[U_a]\,[A' + I]\,[U_a]^{-1} = [U_a]\,[A]\,[U_a]^{-1} = [J + I]$

$[U_b]\,[B' + I]\,[U_b]^{-1} = [U_b]\,[B]\,[U_b]^{-1} = [J + I]$

Thus

____

$*1 \quad [B] = [U_b]^{-1}\,[U_a]\,[A]\,[U_a]^{-1}\,[U_b]$

____

Now it might seem that "maximal rank nilpotent" is a very special case; but in fact, although I haven't proved it as a theorem, from a couple of lines of reasoning it will always be true for the coefficient arrays of Orthogonal Polynomial sequences. In addition finding $[U_a], [U_b]$ is really quite elementary.

If anybody likes I am sure I can demonstrate the carry through of the above to prove:

$$e^{xt}e^{yt} = e^{(x+y)t} \Longrightarrow B_n^{(a+b)} = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) B_k^{(a)}(x)\, B_{n-k}^{(b)}(y)$$

Where t is "creation"/derivative matrix $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{bmatrix}$ and the e^{xt} terms are the coefficient arrays for Binomial polynomials.

Generically for Appell sequences.

This is from "Umbral Calculus" Roman page 94; Generating Function and Sheffer Identity for Bernoulli Polynomials: of order $a+b, a, b$.